INVERSE SPECTRAL PROBLEMS ON A CLOSED MANIFOLD

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Abstract. In this paper we consider two inverse problems on a closed connected Riemannian manifold (M, g). The first one is a direct analog of the Gel'fand inverse boundary spectral problem. To formulate it, assume that M is divided by a hypersurface Σ into two components and we know the eigenvalues λ_j of the Laplace operator on (M, g) and also the Cauchy data, on Σ, of the corresponding eigenfunctions ϕ_j, i.e. $\phi_j|_\Sigma$, $\partial_\nu \phi_j|_\Sigma$, where ν is the normal to Σ. We prove that these data determine (M, g) uniquely, i.e. up to an isometry. In the second problem we are given much less data, namely, λ_j and $\phi_j|_\Sigma$ only. However, if Σ consists of at least two components, Σ_1, Σ_2, we are still able to determine (M, g) assuming some conditions on M and Σ. These conditions are formulated in terms of the spectra of the manifolds with boundary obtained by cutting M along Σ_i, $i = 1, 2$, and are of a generic nature. We consider also some other inverse problems on M related to the above with data which is easier to obtain from measurements than the spectral data described.

1. Introduction and main results

In this paper we consider some inverse spectral problems on a closed connected Riemannian manifold (M, g). The first motivation to consider inverse problems on Riemannian manifolds comes from spectral geometry. The famous problem here, posed by Bochner and formulated by Kac in the paper "Can one hear the shape of a drum?", [14], is the problem of identifiability of the shape of a $2-$dimensional domain from the eigenvalues of its Dirichlet Laplacian. More generally, the question is to find the relations between the spectrum of a Riemannian manifold (M, g), i.e. the spectrum of the Laplace-Beltrami operator $-\Delta_g$ on it, and geometry of this manifold. In particular, one can ask, following Bochner-Kac, if the spectrum of $-\Delta_g$ determines the geometry. However, already in 1966, it was known that, in higher dimensions, the answer to this question is negative. Indeed, in 1964 Milnor [27] found the first counter-example, a pair of Riemannian flat tori of dimension 16 which are isospectral, i.e. the spectra of their Laplacians coincide, but not isometric. As for the original Bochner-Kac problem in dimension 2, the answer was found only in early 90th. Namely, in 1985 Sunada [32] introduced a method of producing examples of non-isometric isospectral compact connected Riemannian manifolds. Although in this paper Sunada did not give the answer to the Bochner-Kac problem, in 1992 Gordon,

Webb and Wolpert [10] extended Sunada's method and settled in the negative this famous problem by constructing two simply connected non-isometric plane domains which are isospectral. Since then there was much activity in this direction with many beautiful mathematical results regarding relations between the spectrum and geometry, see e.g. [11] and [36] for the current state of art in this area.

It is clear from the above that, in order to determine geometry of a closed Riemannian manifold, further spectral information is needed. The nature of this information can be found if we look at inverse boundary problems. In this case, the data given is the trace on ∂M of the resolvent of the Laplacian with, say, Neumann boundary condition. Depending on whether this information is given for one or many values of the spectral parameter, these inverse boundary problems were originally posed by Calderon [7] and Gel'fand [9]. These inverse boundary problems were solved, at least on the level of uniqueness and sometimes conditional stability, for the Laplace-Beltrami operator and also the other types of scalar operators, in e.g. [1]–[6], [16, 18, 19], [23]–[25], [28]–[30], [33, 34] and monographs [13] or [17] with further references therein.

As the first inverse problem considered in this paper is an analog of the Gel'fand inverse boundary problem, we first reformulate the Gel'fand problem in an equivalent form which, however, has more "spectral" flavor. Namely, let λ_j and ϕ_j be the eigenvalues and normalized eigenfunctions of the Laplace operator with Neumann boundary condition,

$$(-\Delta_g - \lambda_j)\phi_j = 0 \quad \text{in} \quad M, \quad \partial_\nu \phi_j|_{\partial M} = 0; \quad (\phi_j, \phi_k)_{L^2(M)} = \delta_{jk}, \tag{1.1}$$

where ∂_ν is the normal derivative to ∂M. Then the Gel'fand problem [9] is the one of the determination of (M, g) from the *boundary spectral data*, i.e. ∂M, $\{\lambda_j, \phi_j|_{\partial M}\}_{j=1}^{\infty}$. Note that, due to the Neumann boundary condition in (1.1), we do actually know the whole Cauchy data $\phi_j|_{\partial M}$, $\partial_\nu \phi_j|_{\partial M}$ of the eigenfunctions on ∂M. To formulate its analog for a closed connected manifold, assume as earlier that we deal with the Laplace operator $-\Delta_g$ on, now closed, Riemannian manifold (M, g). As earlier, we assume that our spectral data are given on an $(m-1)-$dimensional submanifold $\Sigma \subset M$, $\Sigma = \partial S$, where $S \subset M$ is an open set. We note that this type of data is natural for various physical applications when sources and receivers are located over some surface in space rather then are scattered over an $m-$dimensional region or put on, probably remote, boundary of M. Such localization is used e.g. in radars, sonars, and in medical ultrasound imaging when a single antenna array is used to produce the wave and to measure the scattered wave. It is typical also in geosciences/seismology where sources and receivers are often located over the surface of the Earth or an ocean.

Returning to the possible generalizations of the Gel'fand problem to inverse problems on closed manifolds and taking into account that Σ splits M into two manifolds with boundary, $\overline{S}$ and $M \setminus S$, we believe that the most straightforward generalization of the Gel'fand problem is the following:

Assume that we are given the *Cauchy spectral data*,

$$\{\Sigma, (\lambda_j, \phi_j|_\Sigma, \partial_\nu \phi_j|_\Sigma)_{j=1}^\infty\}. \tag{1.2}$$

Does these data determine (M, g)?

Then the first main result of the paper is

Theorem 1.1. *The Cauchy spectral data (1.2) determine the manifold (M, g) up to an isometry.*

Probably a more surprising result concerns with another inverse spectral problem associated with Σ. In this case we have less data, namely, the *Dirichlet spectral data*,

$$\{\Sigma,\ (\lambda_j, \phi_j|_\Sigma)_{j=1}^\infty\}, \tag{1.3}$$

and would like to determine (M, g) from these data. It turns out that this is still possible when the set S consists of two subsets,

$$S = S_1 \cup S_2, \quad \overline{S}_1 \cap \overline{S}_2 = \emptyset,$$

if we assume some generic conditions on S_1, S_2. To state this result, we define the Dirichlet spectrum of the Laplace operator on a manifold $\widetilde{S}$ with boundary to be a collection of all eigenvalues $\lambda \in \mathbb{R}$ of $-\Delta_g$ such that

$$-\Delta_g \phi = \lambda \phi, \quad \text{in } \widetilde{S}, \quad \phi|_{\partial \widetilde{S}} = 0$$

with some non-zero ϕ.

Theorem 1.2. *Assume that the Dirichlet spectra of the Laplace operators on $\overline{S_1}$, $\overline{S_2}$, $M \setminus S$, $M \setminus S_1$ and $M \setminus S_2$ are all disjoint. Then the Dirichlet spectral data determine the manifold up to an isometry.*

The paper is organized as follows. In Section 2 we present some auxiliary results for the transmission problems on M and the metric reconstruction on Σ. Section 3 deals with the reconstruction of the Riemannian manifold (M, g) from the Cauchy spectral data and contains the proof of Theorem 1.1. Section 4 is devoted to the inverse problem with the Dirichlet spectral data and provides the proof of Theorem 1.2. Section 5 contains some further generalizations of the problem and its alternative formulations which may be useful in practical applications.

2. Auxiliary results

2.1. **Transmission problem at fixed frequency.** Consider a closed connected smooth $m-$dimensional Riemannian manifold (M, g) and an open non-empty set $S \subset M$ with smooth $(m-1)-$dimensional boundary $\Sigma := \partial S \neq \emptyset$. Let $u : M \to \mathbb{R}$ be a function such that its restrictions u_+, u_- onto $M_+ = M \setminus S$ and $M_- = \overline{S}$ are H^2-smooth. We define the traces of u on the different sides of Σ by

$$\begin{aligned} \left(u_\pm|_\Sigma\right)(x) &= \lim_{h\to 0^-} u(x \pm h\nu(x)), \quad x \in \Sigma, \\ \left(\partial_\nu u_\pm|_\Sigma\right)(x) &= \lim_{h\to 0^-} \partial_\nu u(x \pm h\nu(x), \quad x \in \Sigma, \end{aligned} \tag{2.1}$$

where $\nu(x)$ is the unit normal to Σ at x pointing towards M_- and the limits are understood in the sense of traces in Sobolev spaces. Denote by $[u] = u_+|_\Sigma - u_-|_\Sigma$ and $[\partial_\nu u] = \partial_\nu u_+|_\Sigma - \partial_\nu u_-|_\Sigma$ the jumps of u and its normal derivative across Σ.

Consider the transmission problem

$$\begin{aligned} &(-\Delta_g - \lambda)u := -g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j u) - \lambda u = 0 \quad \text{in} \quad M \setminus \Sigma, \\ &[u] = f \quad \text{on} \quad \Sigma, \quad [\partial_\nu u] = h \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.2}$$

where $g = \det(g_{ij})$ and $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. Although, in general, problem (2.2) may be considered with $f \in H^{3/2}(\Sigma)$, $h \in H^{1/2}(\Sigma)$, it will be sufficient and convenient for us to take $f, h \in C^\infty(\Sigma)$. Problem (2.2) is equivalent to the problem

$$(-\Delta_g - \lambda)u = h\delta_\Sigma + f\partial_\nu\delta_\Sigma, \tag{2.3}$$

where $h\delta_\Sigma$ and $f\partial_\nu\delta_\Sigma$ are distributions defined as

$$\langle h\delta_\Sigma, \psi\rangle = \int_\Sigma h\psi|_\Sigma \, dS_g, \quad \langle f\partial_\nu\delta_\Sigma, \psi\rangle = -\int_\Sigma f\left(\partial_\nu\psi\right)|_\Sigma \, dS_g,$$

dS_g being the volume element on Σ, for any $\psi \in C^\infty(M)$.

Let $\lambda \notin \sigma(-\Delta_g)$, where $\sigma(-\Delta_g)$ is the spectrum of the Laplace operator $-\Delta_g$. Then problem (2.2) has a unique solution. Moreover, its formulation in form (2.3) makes it possible to represent this solution, $u = u^{f,h}(x, \lambda)$, for $x \in M \setminus \Sigma$, as a sum of a single- and double-layer potentials,

$$\begin{aligned} &u^{f,h}(x,\lambda) = u^{0,h}(x,\lambda) + u^{f,0}(x,\lambda), \\ &u^{0,h}(x,\lambda) = S_\lambda h, \;\; u^{f,0}(x,\lambda) = -D_\lambda f, \\ &S_\lambda h(x) = \int_\Sigma G(x,y;\lambda)h(y)dS_g(y), \quad D_\lambda f(x) = \int_\Sigma \partial_{\nu(y)}G(x,y;\lambda)f(y)dS_g(y), \end{aligned} \tag{2.4}$$

where $G(x, y; \lambda)$ is Green's function,

$$(-\Delta_g - \lambda)G(x, y; \lambda) = \delta_y(x).$$

Note that the single-layer potential is well-defined on and continuous across Σ. Also, $D_\lambda f$ can be continuously extended from $M_\pm^{int}$ to $M_\pm$ and, for $x \in \Sigma$,

$$(D_\lambda f)_\pm|_\Sigma(x) = D_\lambda^o f(x) \pm \frac{1}{2} f(x), \quad D_\lambda^o f(x) := \int_\Sigma \partial_{\nu(y)} G(x, y; \lambda) f(y) dS_g(y). \tag{2.5}$$

Similar, the normal derivative of $S_\lambda h$ is continuously extended, from the left and right, to $x \in \Sigma$,

$$\partial_\nu (S_\lambda h)_\pm|_\Sigma(x) = J_\lambda h(x) \mp \frac{1}{2} h(x), \quad J_\lambda h(x) := \int_\Sigma \partial_{\nu(x)} G(x, y; \lambda) h(y) dS_g(y), \tag{2.6}$$

where the integrals in (2.5)-(2.6) are weakly singular ones. Also integrals in (2.5)-(2.6) are mutually adjoint on Σ in the sense of duality

$$\begin{aligned} \langle D_\lambda^o f,\, h\rangle &:= \int_\Sigma \left(D_\lambda^o f\right)(x) h(x)\, dS_g(x) \qquad\qquad (2.7)\\ = \quad \langle f,\, J_\lambda h\rangle &:= \int_\Sigma f(x) \left(J_\lambda h\right)(x)\, dS_g(x), \quad f, h \in C^\infty(\Sigma). \end{aligned}$$

(For these results see e.g. [8]. Note that, due to the local nature of constructions in [8], they are valid not only for the Euclidian case considered in there but also for manifolds.)

2.2. **Transmission problem in time domain.** In this subsection we consider the transmission problem for the wave equation associated with the Laplace operator, $-\Delta_g$,

$$\begin{aligned} &(\partial_t^2 - \Delta_g) u^{f,h} = 0 \quad \text{in } (M \setminus \Sigma) \times \mathbb{R}, \\ &[u^{f,h}] = f \quad \text{on } \Sigma \times \mathbb{R}, \quad [\partial_\nu u^{f,h}] = h \quad \text{on } \Sigma \times \mathbb{R}, \\ &u^{f,h}|_{t < \min(t_h, t_f)} = 0, \end{aligned} \tag{2.8}$$

where $h, f \in C_+^\infty(\Sigma \times \mathbb{R})$. This space consists of $C^\infty-$ smooth functions equal to 0 for sufficiently large negative t, i.e.

$$h = 0 \quad \text{for } t < t_h, \quad f = 0 \quad \text{for } t < t_f.$$

The results obtained below will be instrumental, although for the special case $f = 0$, in section 4. However, as considerations for $f = 0$ and $f \neq 0$ are parallel, we will consider the general case.

In lemma below, we use the following spectral decomposition

$$u^{f,h}(x, t) = \sum_{j=1}^\infty u_j^{f,h}(t) \phi_j(x). \tag{2.9}$$

Lemma 2.1. *(Blagovestchenskii identity) Given the Cauchy spectral data (1.2) it is possible to evaluate the Fourier coefficients $u_j^{0,h}(t)$, $u_j^{f,0}(t)$ of the waves $u^{0,h}(x,t)$, $u^{f,0}(x,t)$, namely,*

$$u_j^{0,h}(t) = \int_{t_h}^{t} \frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} h_j(s)ds, \quad h_j(t) := \int_\Sigma h(\cdot,t)\phi_j|_\Sigma dS_g, \tag{2.10}$$

$$u_j^{f,0}(t) = -\int_{t_f}^{t} \frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} f_j(s)ds, \quad f_j(t) := \int_\Sigma f(\cdot,t)\partial_\nu\phi_j|_\Sigma dS_g,$$

and, for $\lambda_1 = 0$, we should substitute $\frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}}$ by $(t-s)$. Moreover,

$$u^{0,h}(x,t) \in C_+^\infty(\mathbb{R}; H^1(M)), \quad u^{f,0}(x,t) \in C_+^\infty(\mathbb{R}; L^2(M)).$$

Proof. We consider first $u^{0,h}$. We have

$$u_j^{0,h}(t) = \left(u^{0,h}(x,t), \phi_j(x)\right)_{L^2(M)} = \int_{M_+} u^{0,h}(x,t)\phi_j(x)dV_g + \int_{M_-} u^{0,h}(x,t)\phi_j(x)dV_g.$$

Thus, (2.8) implies that,

$$\partial_t^2 u_j^{0,h}(t) = \int_{M_+} \Delta_g u^{0,h}(x,t)\phi_j(x)dV_g + \int_{M_-} \Delta_g u^{0,h}(x,t)\phi_j(x)dV_g =$$
$$\int_\Sigma \left(\partial_\nu u_+^{0,h} - \partial_\nu u_-^{0,h}\right)|_\Sigma \phi_j|_\Sigma dS_g - \int_\Sigma \left(u_+^{0,h} - u_-^{0,h}\right)|_\Sigma \partial_\nu\phi_j|_\Sigma dS_g$$
$$+ \int_M u^{0,h}(x,t)\Delta_g\phi_j(x)dV_g = h_j(t) - \lambda_j u_j^{0,h}(t),$$

where we use that $\left(\partial_\nu u_+^{0,h} - \partial_\nu u_-^{0,h}\right)|_\Sigma = h$, $\left(u_+^{0,h} - u_-^{0,h}\right)|_\Sigma = 0$. Solving this second order ordinary differential equation together with the initial conditions $u_j^{0,h}(t_h) = 0$, $\partial_t u_j^{0,h}(t_h) = 0$, provides the first formula in (2.10). Similar considerations provide the second formula in (2.10).

To prove the second part of Lemma, we rewrite problem (2.8) in the form, cf. (2.3),

$$\partial_t^2 u^{0,h} - \Delta_g u^{0,h} = h\delta_\Sigma, \quad u^{0,h}|_{t<t_h} = 0.$$

As, for any t, $h(\cdot,t)\delta_\Sigma \in H^{-1}(M)$, we have, for any $k = 0, 1, \dots,$ that

$$\sum_{j=1}^\infty (\lambda_j+1)^{-1}|\partial_t^k h_j(t)|^2 \in C_+^\infty(\mathbb{R}). \tag{2.11}$$

On the other hand, representation (2.10) yields that, for $j \geq 2$,

$$\partial_t^k u_j^{0,h}(t) = \int_0^{t-t_h} \frac{\sin(\sqrt{\lambda_j}s)}{\sqrt{\lambda_j}} \partial_t^k h_j(t-s)ds.$$

This implies that

$$\partial_t^k u_j^{0,h}(t) = \frac{1}{\lambda_j}\partial_t^k h_j(t) - \frac{1}{\lambda_j}\int_{t_h}^t \cos(\sqrt{\lambda_j}(t-s))\partial_s^{k+1} h_j(s)ds,$$

with obvious modification for $j=1$. Thus,

$$\sum_{j=1}^{\infty}(\lambda_j+1)|\partial_t^k u_j^{0,h}(t)|^2 \le C\sum_{j=1}^{\infty}(\lambda_j+1)^{-1}|\partial_t^k h_j(t)|^2$$

$$+C'(t-t_h)\int_{t_h}^t \sum_{j=1}^{\infty}(\lambda_j+1)^{-1}|\partial_s^{k+1}h_j(s)|^2\, ds.$$

This inequality, together with (2.11), implies the desired result for $u^{0,h}(x,t)$. Similarly, we prove that $u^{f,0}(x,t) \in C_+^\infty(\mathbb{R}; L^2(M))$. □

Lemma 2.1 immediately implies the following result:

Corollary 2.2. *The Dirichlet spectral data (1.3) makes it possible to find the trace on $\Sigma\times\mathbb{R}$, i.e. $u^{0,h}|_{\Sigma\times\mathbb{R}}$, for any $h \in C_+^\infty(\Sigma\times\mathbb{R})$.*

Proof. The result follows from formula (2.10) taking into the account that, due to $u^{0,h} \in C_+^\infty(\mathbb{R};\, H^1(M))$, the series (2.9) converges, for any t, in $H^1(M)$. Therefore, the trace

$$u^{0,h}|_{\Sigma\times\mathbb{R}}(x,t) = \sum_{j=1}^{\infty} u_j^{0,h}(t)\phi_j|_\Sigma(x),$$

where the right-hand side converges in $C_+^\infty(\mathbb{R};\, H^{1/2}(\Sigma))$ □

Let us now warm up by considering a simple inverse problem when the part M_- of the manifold M is known and show how Lemma 2.1 and Corollary 2.2 can be used to recover M_+.

Example 2.1. Assume that, in addition to (1.3), we know the manifold $(\overline{M}_-, g|_{\overline{M}_-})$. Then the manifold $(M_+, g|_{M_+})$ can be recovered up to an isometry.

Indeed, let $h \in C_+^\infty(\Sigma\times\mathbb{R}_+)$, i.e. $h \in C_+^\infty(\Sigma\times\mathbb{R})$ and $t_h>0$. Using Corollary 2.2, we can find $u^{0,h}|_{\Sigma\times\mathbb{R}_+}$.

Consider now the Dirichlet initial-boundary value problem in $M_-\times\mathbb{R}_+$ with Dirichlet data being $u^{0,h}|_{\Sigma\times\mathbb{R}_+}$. As $(M_-,\, g|_{M_-})$ is known, we can then find $u^{0,h}|_{M_-\times\mathbb{R}_+}$ and, therefore, the normal derivative $\partial_\nu u_-^{0,h}|_{\Sigma\times\mathbb{R}}$. Thus, we can find $\partial_\nu u_+^{0,h}|_{\Sigma\times\mathbb{R}} = \partial_\nu u_-^{0,h}|_{\Sigma\times\mathbb{R}} + h$.

It is shown in the proof of Lemma 4.4 below that, when h runs over $C_+^\infty(\Sigma\times\mathbb{R}_+)$, then $u^{0,h}|_{\Sigma\times\mathbb{R}_+}$ also runs over the whole $C_+^\infty(\Sigma\times\mathbb{R}_+)$. Therefore, the set of pairs,

$$\{(u^{0,h}|_{\Sigma\times\mathbb{R}}, \partial_\nu u_+^{0,h}|_{\Sigma\times\mathbb{R}})\, :\, h\in C_+^\infty(\Sigma\times\mathbb{R})\},$$

defines the graph of the non-stationary Dirichlet-to-Neumann map for M_+.

It is, however, shown in e.g. [17] that this map determines $(M_+, g|_{M_+})$ upto an isometry.

2.3. **Metric on Σ.** We complete this section showing that the Dirichlet spectral data determines the metric on Σ. This result will be needed later, in sections 3 and 4.

Lemma 2.3. *The Dirichlet spectral data (1.3) determine the distance function on Σ and, therefore, the inherited metric tensor, $g|_\Sigma$, on Σ.*

Proof. Let $H(x,y;t)$ be the heat kernel,

$$\partial_t H(x,y;t) - \Delta_g H(x,y;t) = \delta_y(x)\delta(t), \quad x, y \in M; \quad H|_{t=0} = 0.$$

Then, uniformly in $M \times M$,

$$t \log H(x,y;t) \to -\frac{1}{4} d_M(x,y),$$

as $t \to 0$, see e.g. [35]. On the other hand, for $t > 0$,

$$H(x,y;t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y),$$

where the convergence takes place in $C^\infty(M \times M \times (0,\infty))$. Thus, we can determine the distance $d_M(x,y)$ between any points $x, y \in \Sigma$ using the Dirichlet spectral data (1.3). Then the distance along Σ is given by

$$d_\Sigma(x,y) = \lim_{\varepsilon \to 0} \inf \sum_{j=0}^{N-1} d_M(z_j, z_{j+1}),$$

where the infimum is taken over all finite sequences of points $z_0, z_1, \ldots, z_N \in \Sigma$ such that $z_0 = x$, $z_N = y$ and $d_M(z_j, z_{j+1}) \le \varepsilon$. Having at hand $d_\Sigma(x,y)$, we can determine the inherited metric tensor, $g|_\Sigma$ on Σ. $\square$

3. Inverse problem with Cauchy spectral data. Theorem 1.1.

In this section we develop a procedure to reconstruct the Riemannian manifold (M,g) from the *Cauchy spectral data* (1.2).

3.1. **From Cauchy spectral data to the response operator.** Let us consider the transmission problem (2.2) with $\lambda \notin \sigma(-\Delta_g)$. Denoting, as in section 2, its solution by $u^{f,h}(x,\lambda)$, we define the *response operator* by setting

$$R_\lambda(f,h) = u_+^{f,h}(\lambda)|_\Sigma.$$

Lemma 3.1. *The Cauchy spectral data (1.2) determine the response operator R_λ for all $\lambda \notin \sigma(-\Delta_g)$.*

Proof. Note that Green's function has the following spectral representation

$$G(x,y;\lambda) = \sum_{j=1}^{\infty} \frac{\phi_j(x)\phi_j(y)}{\lambda - \lambda_j}, \tag{3.1}$$

where the sum converges the sense of operators in $L^2(M)$ and we assume, without loss of generality, that the eigenfunctions ϕ_j are real-valued. Thus, given the Cauchy spectral data and using (3.1), one can formally construct the single- and double-layer potentials on Σ,

$$S_\lambda h(x)|_\Sigma \underset{\text{formally}}{=} \int_\Sigma \big[\sum_{j=1}^{\infty} \frac{\phi_j(x)|_\Sigma \phi_j(y)}{\lambda - \lambda_j}\big] h(y) dS_g(y), \tag{3.2}$$

$$D^o_\lambda f(x) \underset{\text{formally}}{=} \int_\Sigma \big[\sum_{j=1}^{\infty} \frac{\phi_j(x)|_\Sigma \partial_{\nu(y)}\phi_j(y)}{\lambda - \lambda_j}\big] f(y) dS_g(y). \tag{3.3}$$

Since

$$R_\lambda(f,h) = S_\lambda h - (\frac{1}{2} + D^o_\lambda) f,$$

it looks that data (1.2) directly determines R_λ. However, we face the difficulty that series (3.1) does not converge pointwise. To deal with this difficulty, consider first the case when $f = 0$. To determine the coefficients in the Fourier expansion

$$u^{0,h}(x,\,\lambda) = \sum_{j=1}^{\infty} (u^{0,h}(\lambda), \phi_j)_{L^2(M)} \phi_j(x), \tag{3.4}$$

we use Green's formula to get

$$(u^{0,h}, \phi_j)_{L^2(M)} = -\frac{1}{\lambda - \lambda_j}(\int_{M_+} + \int_{M_-})(\Delta_g u^{0,h}\phi_j - u^{0,h}\Delta_g \phi_j) dV_g = \tag{3.5}$$
$$-\frac{1}{\lambda - \lambda_j}\int_\Sigma (\partial_\nu u^{0,h}_+ \phi_j|_\Sigma - \partial_\nu u^{0,h}_- \phi_j|_\Sigma - u^{0,h}_+ \partial_\nu \phi_j|_\Sigma + u^{0,h}_- \partial_\nu \phi_j|_\Sigma) dS_g = -\frac{1}{\lambda - \lambda_j}\int_\Sigma h\phi_j|_\Sigma dS_g.$$

Since the series (3.4), (3.5) converges in $H^1(M)$, so that the trace is given by

$$u^{0,h}(x,\lambda)|_\Sigma = -\sum_{j=1}^{\infty} \big[\int_\Sigma \phi_j(y) h(y) dS_g(y)\big] \frac{\phi_j(x)|_\Sigma}{\lambda - \lambda_j},$$

where the series converges in $H^{1/2}(\Sigma)$. Hence (3.2) is well-defined. To compute (3.2) we also need to know the Riemannian volume $dS_g(x)$ of Σ. By Lemma 2.3, it can be found from data (1.2).

Let us now show that (3.3) is well-defined. First note that $u^{0,h}|_{M_+} \in H^1(M_+) \cap H^2_{loc}(M_+)$. Thus, we can define, for any $\varepsilon > 0$, the normal derivative $\partial_\nu u^{0,h}|_{\Sigma^\varepsilon}$, where $\Sigma^\varepsilon = \{x \in M_+ : d(x,\Sigma) = \varepsilon\}$. Clearly, for any $\Psi \in H^1(M_+)$,

$$\begin{aligned}\int_{\Sigma^\varepsilon} (\partial_\nu u^{0,h}|_{\Sigma^\varepsilon})\Psi|_{\Sigma^\varepsilon}\, dS_g &= \int_{M_+^\varepsilon} \Delta_g u^{0,h}\, \Psi\, dV_g + \int_{M_+^\varepsilon} (\nabla_g u^{0,h},\, \nabla_g \Psi)_g\, dV_g \\ &= -\lambda \int_{M_+^\varepsilon} u^{0,h}\, \Psi\, dV_g + \int_{M_+^\varepsilon} (\nabla_g u^{0,h},\, \nabla_g \Psi)_g\, dV_g \qquad (3.6)\end{aligned}$$

where $M_+^\varepsilon = \{x \in M_+ : d(x,\Sigma) \geq \varepsilon\}$ and we have used equation (2.2). As the right-hand side of (3.6) has a limit, when $\varepsilon \to 0$, and $\Psi|_\Sigma$ runs over $H^{1/2}(\Sigma)$ when Ψ runs over $H^1(M_+)$, this defines $\partial_\nu u_+^{0,h}|_\Sigma$. As we can choose, for any $\Psi|_\Sigma \in H^{1/2}(\Sigma)$ its extension Ψ so that $\|\Psi\|_{H^1} \leq C\|\Psi|_\Sigma\|_{H^{1/2}}$,

$$\|\partial_\nu u^{0,h}(\lambda)_+|_\Sigma\|_{H^{-1/2}(\Sigma)} \leq C\|u^{0,h}(\lambda)|_{M_+}\|_{H^1(M_+)}. \qquad (3.7)$$

Define the sources-to-Dirichlet operator by setting

$$J_\lambda h = \partial_\nu u^{0,h}(\lambda)_+|_\Sigma,$$

see (2.6). Taking λ-derivative of (3.5), we get

$$\frac{\partial}{\partial\lambda} u^{0,h}(x,\lambda) = \sum_{j=1}^\infty \Big[\int_\Sigma \phi_j(y)h(y)\, dS_g(y)\Big]\frac{\phi_j(x)}{(\lambda_j-\lambda)^2} = (-\Delta_g - \lambda)^{-1} u^{0,h}(\lambda)$$

that converges in $H^3(M)$. So we have a well-defined object

$$\frac{\partial}{\partial\lambda}\partial_\nu u^{0,h}(\lambda)|_\Sigma = \sum_{j=1}^\infty \Big[\int_\Sigma \phi_j(y)h(y)\, dS_g(y)\Big]\frac{\partial_{\nu(x)}\phi_j(x)|_\Sigma}{(\lambda_j-\lambda)^2},$$

where the convergence holds in $H^{3/2}(\Sigma)$. As $\frac{\partial}{\partial\lambda}(J_\lambda h) = \frac{\partial}{\partial\lambda}\partial_\nu u^{0,h}(\lambda)|_\Sigma$, we can compute, for any $h \in C^\infty(\Sigma)$, $\frac{\partial}{\partial\lambda}(J_\lambda h)$ using the Cauchy spectral data of Σ.

Let $\lambda \in \mathbb{R}$, $\lambda \neq \lambda_j$, and let $\gamma_T \subset \mathbb{C}$ be the line segment from λ to iT. As

$$J_\lambda h = \int_{iT}^\lambda \frac{\partial}{\partial\tau}(J_\tau h)\, d\tau + J_{iT}h,$$

we have

$$J_\lambda h = \lim_{T\to\infty}\Big(\int_{iT}^\lambda \frac{\partial}{\partial\tau}(J_\tau h)\, d\tau + J_{iT}h\Big).$$

By Lemma 3.2 below, we get $\lim_{T\to\infty} J_{iT}h = \lim_{T\to\infty}\partial_\nu u^{0,h}(iT)_+|_\Sigma = \frac{1}{2}h$. This implies that

$$J_\lambda h = \frac{1}{2}h + \lim_{T\to\infty}\int_{iT}^\lambda \frac{\partial}{\partial\tau}(J_\tau h)\, d\tau,$$

where the right-hand side can be computed using the Cauchy spectral data.

To complete the proof, we recall, see equation (2.7), that J_λ is adjoint of D_λ^0. Thus we can find $D^o_\lambda f$ using the Cauchy spectral data. $\square$

In the proof of the above Lemma we used the following asymptotics, with respect to singularity, of $u^{0,h}$ near Σ.

Lemma 3.2. *Let $\lambda \in \mathbb{C}$, $Im\lambda > 0$ and $\mathrm{Re}\sqrt{-\lambda} < 0$. Then uniformly for $|\arg(\lambda)| \geq \delta$,*

$$u^{0,h}(\lambda)_\pm|_\Sigma \to 0, \quad \partial_\nu u^{0,h}(\lambda)_\pm|_\Sigma \to \pm\frac{1}{2}h, \quad \text{as } \lambda \to \infty, \tag{3.8}$$

in $H^{3/2}(\Sigma)$, $H^{1/2}(\Sigma)$, correspondingly.

Proof. Let us first fix local coordinates on M near Σ, $x = (x', x^m)$, where $x' = (x^1, \dots, x^{m-1})$ are some local coordinates on Σ and x^m is the signed distance to Σ,

$$x^m = \pm\mathrm{dist}(x, \Sigma) \quad \text{for } x \in M_\pm.$$

In these coordinates, we introduce

$$v^{0,h}_\pm(x;\lambda) := \begin{cases} \frac{h(x')}{2\sqrt{-\lambda}} e^{x^m\sqrt{-\lambda}}\zeta(x^m), & x \in M_+, \\ \frac{h(x')}{2\sqrt{-\lambda}} e^{-x^m\sqrt{-\lambda}}\zeta(x^m), & x \in M_-, \end{cases} \tag{3.9}$$

where $\zeta(x^m)$ is a smooth cut-off function equal to 1 near $x^m = 0$ supported in $(-a, a)$ with sufficiently small $a > 0$. Outside the a-neighborhood of Σ, the functions $v_\pm$ are defined to be zero. Writing

$$\Delta_g = \partial^2_{x^m} + p(x)\partial_{x^m} + Q(x, \partial_{x^1}, \dots, \partial_{x^{m-1}})$$

in the above coordinates and using the fact that

$$\int_0^a |e^{x^m\sqrt{-\lambda}}\zeta(x^m)|^2 dx^m \leq C(\sqrt{-\lambda})^{-1},$$

we see that (2.3), (3.9) yield

$$u^{0,h}(x,\lambda) = v^{0,h}(x,\lambda) + w^{0,h}(x,\lambda),$$

where $w^{0,h} \in H^2(M)$ satisfies

$$(-\Delta_g - \lambda)w^{0,h} = H^h(\lambda), \quad \|H^h(\lambda)\|_{L^2(M)} \leq C_h(1 + |\lambda|)^{-1/4}.$$

As $\|(-\Delta_g - \lambda)^{-1}\| \leq \mathrm{dist}(\lambda, \sigma(-\Delta_g))$, where the norm is the operator norm in $L^2(M)$, this implies

$$\|w^{0,h}(\lambda)\|_{H^2(M)} \leq C_{h,\delta}|\lambda|^{-1/4},$$

when $|\arg(\lambda)| \geq \delta > 0$ and $|\lambda| > 1$. Combining this estimate with (3.9), we see (3.8).

$\square$

Remark 3.1. Analyzing the behaviour of $u^{f,h}$ near $\Sigma \times \mathbb{R}$, we can show, cf. considerations leading to (3.8) and (3.9), that

$$u^{f,h}(x,t)|_{M_\pm \times \mathbb{R}} \in C_+^\infty(M_\pm \times \mathbb{R}), \tag{3.10}$$

meaning that $u^{f,h}(x,t)|_{M_\pm^{int} \times \mathbb{R}}$ may be continued to $M_\pm \times \mathbb{R}$ to satisfy (3.10).

3.2. **Reconstruction of the manifold using the response operator.** Recall, see e.g. [17], that, if (N,g), $\partial N \neq \emptyset$, then its Neumann-to-Dirichlet operators, $\Lambda_\lambda(N)$ are defined as

$$\Lambda_\lambda(N)\psi = w^\psi(\lambda)|_{\partial N},$$

where $w^\psi(x,\lambda)$ is the solution to the Neumann problem

$$-\Delta_g w^\psi(x,\lambda) = \lambda w^\psi(x,\lambda), \ x \in N^{int}, \quad \partial_\nu w^\psi(x,\lambda)|_{\partial N} = \psi,$$

for $\lambda \notin \sigma(-\Delta_g^N)$, $\sigma(-\Delta_g^N)$ being the spectrum of the Neumann Laplacian on N.

Lemma 3.3. *Given the Cauchy spectral data (1.2) it is possible to find the Neumann-to-Dirichlet operators $\Lambda_\lambda(M_\pm)$ for $\lambda \notin \sigma(-\Delta_\pm^N)$, where $-\Delta_\pm^N$ stands for the Neumann Laplacian on $M_\pm$.*

Proof. We start with $\Lambda_\lambda(M_-)$, assuming $\lambda \notin \big(\sigma(-\Delta_g) \cup \sigma(-\Delta_-^N) \cup \sigma(-\Delta_+^D)\big)$, where $-\Delta_\pm^D$ is the Dirichlet Laplacian in $M_\pm$, correspondingly. Then, for any $h \in C^\infty(\Sigma)$, there is a unique solution, $w_-^h(x,\lambda) \in C^\infty(M_-)$, satisfying

$$-\Delta_g w_-^h(x,\lambda) = \lambda w_-^h(x,\lambda) \text{ in } M_-, \quad \partial_\nu w_-^h(\cdot,\lambda)|_\Sigma = -h,$$

where, as in equation (2.1), ν is the unit normal pointing towards M_-.

Consider

$$w^h(x,\lambda) = \begin{cases} 0 & \text{in} \quad M_+, \\ w_-^h(x,\lambda) & \text{in} \quad M_-. \end{cases}$$

Clearly, $w^h(x,\lambda)$ solves (2.2) with $[w^h] := f = -w_-^h(\cdot,\lambda)|_\Sigma$, $[\partial_\nu w^h] = h$. Moreover, with this f and h,

$$R_\lambda(f,h) = 0. \tag{3.11}$$

These considerations show that, for any h, there is f such that (3.11) is satisfied and we can consider (3.11) as an equation for f when h is given. Let us show that the solution to (3.11) is unique if $\lambda \notin \big(\sigma(-\Delta_-^N) \cup \sigma(-\Delta_+^D)\big)$. This will allow us to uniquely define $f = f^h(\lambda) = -w_-^h(\cdot,\lambda)|_\Sigma$ as the solution to (3.11). Then,

$$\Lambda_\lambda(M_-)h = -f^h(\lambda).$$

To prove uniqueness, assume that there is f such that

$$R_\lambda(f,0) = 0.$$

As $\lambda \notin \sigma(-\Delta_+^D)$, this implies that $u_+^{f,0}(x,\lambda) = 0$. As $[\partial_\nu u^{f,0}] = 0$, we see that

$$\partial_\nu u_-^{f,0}(\cdot,\lambda)|_\Sigma = 0.$$

However, $\lambda \notin \sigma(-\Delta_-^N)$, so that $u_-^{f,0}(x,\lambda) = 0$, i.e. $f = 0$.

Combining with Lemma 3.1, we see that the Cauchy spectral data (1.2) determine $\Lambda_\lambda(M_-)$ for $\lambda \notin \big(\sigma(-\Delta_g) \cup \sigma(-\Delta_-^N) \cup \sigma(-\Delta_+^D)\big)$. Since $\Lambda_\lambda(M_-)$ is a meromorphic operator-valued function with simple poles at $\sigma(-\Delta_-^N)$, this determines $\Lambda_\lambda(M_-)$ uniquely.

As $u_-^{f,h}|_\Sigma = R_\lambda(f,h) - f$, we can repeat the previous arguments for $\Lambda_\lambda(M_+)$. $\square$

Theorem 1.1 follows from Lemma 3.3 taking into account that $\Lambda_\lambda(M_\pm)$ determine $(M_\pm, g_\pm)$ up to an isometry, see [17], section 4.1. Thus to recover (M,g) we should just glue (M_-, g_-) and (M_+, g_+) along given Σ.

4. Inverse problem with Dirichlet spectral data. Theorem 1.2.

In this section, we will develop a procedure to reconstruct the Riemannian manifold (M,g) from the *Dirichlet spectral data* (1.3). We will assume that $S \subset M$ consists of two open subsets S_1, S_2, $\overline{S_1} \cap \overline{S_1} = \emptyset$, $S = S_1 \cup S_2$. As in section 3, we assume that $\Sigma := \partial S = \Sigma_1 \cup \Sigma_2$, $\Sigma_i = \partial S_i, i = 1,2$, are smooth. Moreover, we assume that the spectra $\sigma(-\Delta^D(\overline{S_i}))$, $\sigma(-\Delta^D(M \setminus S_i)), i = 1,2$, and $\sigma(-\Delta^D(M \setminus S))$ are all disjoint.

4.1. **An approximate controllability result.** Consider the following transmission problem

$$\begin{aligned} &(\partial_t^2 - \Delta_g)u = 0, \quad \text{in } (M \setminus \Sigma) \times \mathbb{R}, \\ &[u]_\Sigma = 0, \quad [\partial_\nu u]_\Sigma = h \in C_+^\infty(\Sigma \times \mathbb{R}), \\ &u|_{t<t_h} = 0, \end{aligned} \tag{4.1}$$

and denote by $u(x,t) = u^{0,h}(x,t)$ its solution. Note that problem (4.1) coincides with problem (2.8) with $f = 0$.

By Lemma 2.1, $u^{0,h} \in C_+^\infty(\mathbb{R}; H^1(M))$ and we can define an operator

$$W : C_+^\infty(\Sigma \times \mathbb{R}) \to H^1(M), \quad Wh := u^{0,h}(0),$$

which is called the wave operator associated with problem (4.1).

Theorem 4.1. *Let $\sigma(-\Delta^D(\overline{S})) \cap \sigma(-\Delta^D(M \setminus S)) = \emptyset$. Then the set*

$$Y = \{Wh : \; h \in C_+^\infty(\Sigma \times \mathbb{R})\} \tag{4.2}$$

is dense in $H^1(M)$.

Proof. Assume that $\psi \in (H^1(M))' = H^{-1}(M)$ is orthogonal to Y,

$$(u^{0,h}(0), \psi)_{H^1(M) \times H^{-1}(M)} = 0 \tag{4.3}$$

for all $h \in C_+^\infty(\Sigma \times \mathbb{R})$. Let e be the solution to the problem,

$$\begin{aligned} & e_{tt} - \Delta_g e = 0, \quad \text{ in } M \times \mathbb{R}, \\ & e|_{t=0} = 0, \ e_t|_{t=0} = \psi. \end{aligned} \tag{4.4}$$

Then similar considerations to those at the end of proof of Lemma 2.1 show that the weak solution, $e(x,t)$ of (4.4) satisfies,

$$e(x,t) = \sum_{j=2}^{\infty} \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} (\psi, \phi_j)_{H^{-1}(M)\times H^1(M)} \phi_j + t(\psi, \phi_1)_{H^{-1}(M)\times H^1(M)} \phi_1 \tag{4.5}$$

and $e \in C(\mathbb{R}; L^2(M)) \cap C^1(\mathbb{R}; H^{-1}(M))$. Observe that, as $\|\phi_j|_\Sigma\|_{H^{1/2}} \le C(\lambda_j + 1)^{1/2}$ and $\lambda_j > Cj^{2/m}$, e has a well-defined trace, in $\mathcal{S}'(\Sigma \times \mathbb{R})$, on $\Sigma \times \mathbb{R}$ with

$$\sum_{j=2}^{J} \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} (\psi, \phi_j) \phi_j|_\Sigma + t(\psi, \phi_1)\phi_1|_\Sigma \to e(x,t)|_{\Sigma\times\mathbb{R}}, \quad \text{as } J \to \infty, \tag{4.6}$$

in $\mathcal{S}'(\Sigma \times \mathbb{R})$.

Let us show that $e(x,t)|_{\Sigma\times\mathbb{R}} = 0$. Choosing $h \in C_0^\infty(\Sigma \times \mathbb{R}_-)$ and using Green's formula, we obtain from (4.1)–(4.4) that

$$\begin{aligned} 0 &= \int_{M\times\mathbb{R}_-} [u^{0,h}(e_{tt} - \Delta_g e) - (u^{0,h}_{tt} - \Delta_g u^{0,h})e] \, dV_g \, dt \\ &= (u^{0,h}(T), \psi)_{H^1(M)\times H^{-1}(M)} + \int_{\Sigma\times\mathbb{R}_-} h\, e \, dS_g \, dt = \int_{\Sigma\times\mathbb{R}_-} h \, e\, dS_g\, dt. \end{aligned}$$

This yields that $e|_{\Sigma\times(-\infty,0)} = 0$. As by (4.4) $e(x,s) = -e(x,-s)$, we see that

$$\operatorname{supp}(e|_{\Sigma\times\mathbb{R}}) = \Sigma \times \{0\}. \tag{4.7}$$

Next we show that

$$e|_{\Sigma\times(-1,1)} \in \widetilde{H}^{-1/2}(\Sigma \times (-1,1)) := \left(H^{1/2}(\Sigma \times (-1,1))\right)'. \tag{4.8}$$

Let X be a local, near Σ, vector field on $\overline{S}$ such that $X|_\Sigma = \partial_\nu|_\Sigma$. Let $h \in H^{1/2}(\Sigma \times (-1,1))$ and $H \in H^1(S \times (-1,1))$ be its continuation into $S \times (-1,1)$, such that $H = 0$ outside the domain of definition of X and

$$\|H\|_{H^1} \le C\|h\|_{H^{1/2}}.$$

Denote by E the primitive, with respect to t, of e in $S \times (-1,1)$,

$$E(x,t) = -\sum_{j=2}^{\infty} \frac{\cos(\sqrt{\lambda_j} t)}{\lambda_j} (\psi, \phi_j)\phi_j(x) + \frac{t^2}{2}(\psi, \phi_1)\phi_1(x), \tag{4.9}$$

$E \in C(\mathbb{R}; H^1(S)) \cap C^1(\mathbb{R}; L^2(S))$. Integrating by parts, we get

$$\int_{\Sigma\times(-1,1)} h\, e\, dS_g\, dt = \int_{S\times(-1,1)} (H\cdot Xe - X^c H\cdot e)\, dV_g\, dt \tag{4.10}$$
$$= \int_S \left([H\cdot XE]|_{t=1} - [H\cdot XE]|_{t=-1}\right) dV_g - \int_{S\times(-1,1)} (\partial_t H\cdot XE + X^c H\cdot e)\, dV_g\, dt,$$

where X^c is the first-order operator adjoint to X. By (4.5), (4.9), the right-hand side of (4.10) can be estimated by

$$C\|H\|_{H^1(S\times(-1,1))} \leq C'\|h\|_{H^{1/2}(\Sigma\times(-1,1))}.$$

Thus the left-hand side of (4.10) is bounded for any $h \in H^{1/2}(\Sigma\times(-1,1))$, proving (4.8).

Now (4.7) implies that $e(x,t)|_{\Sigma\times\mathbb{R}} = \sum_{i=0}^{I} e_i(x)\partial^i\delta(t)$, with some finite I, see e.g. [12, ex. 5.1.2]. Thus, (4.8) yields

$$e(x,t)|_{\Sigma\times\mathbb{R}} = 0. \tag{4.11}$$

The last step of the proof is to show that this equation yields that $e = 0$ in $M\times\mathbb{R}$. Using relation (4.6), equation (4.11) and making the partial Fourier transform, $t \to k$, we see that the distribution, $\widehat{e}(x,k) \in \mathcal{S}'(\Sigma\times\mathbb{R})$, satisfies

$$\begin{aligned} \widehat{e}(x,k) \;\; = \;\; & i\Big(\frac{1}{2}\sum_{j=2}^{\infty}\frac{\delta(k-\sqrt{\lambda_j})}{\sqrt{\lambda_j}}(\psi,\phi_j)\phi_j|_\Sigma \\ & +\delta'(k)(\psi,\phi_1)\phi_1|_\Sigma - \frac{1}{2}\sum_{j=2}^{\infty}\frac{\delta(k+\sqrt{\lambda_j})}{\sqrt{\lambda_j}}(\psi,\phi_j)\phi_j|_\Sigma\Big) = 0. \end{aligned}$$

This implies that $(\psi,\phi_1)_{H^{-1}(M)\times H^1(M)} = 0$ and, for any $\widetilde{j} = 2,\dots,$

$$\sum_{\lambda_j=\lambda_{\widetilde{j}}} (\psi,\phi_j)_{H^{-1}(M)\times H^1(M)}\phi_j|_\Sigma = 0, \tag{4.12}$$

where the last sum takes into account eigenspaces of an arbitrary multiplicity. Consider the function

$$\Phi(x) := \sum_{\lambda_j=\lambda_{\widetilde{j}}} (\psi,\phi_j)_{H^{-1}(M)\times H^1(M)}\phi_j(x), \quad x\in M.$$

It satisfies the Dirichlet boundary condition, $\Phi|_\Sigma = 0$, see (4.12), and, as ϕ_j are eigenfunctions of $-\Delta_g$ with $\lambda_j = \lambda_{\widetilde{j}}$, the equation

$$-\Delta_g\Phi(x) = \lambda_{\widetilde{j}}\Phi(x), \quad x\in M.$$

Thus $\Phi|_S$ is an eigenfunction of $-\Delta^D(S)$, while $\Phi|_{M\setminus S}$ is an eigenfunction of $-\Delta^D(M\setminus S)$. However, as $\sigma(-\Delta^D(\overline{S}))\cap\sigma(-\Delta^D(M\setminus S)) = \emptyset$, we have that $\Phi|_{\overline{S}} = 0$ or $\Phi|_{M\setminus S} = 0$. In any case, by the uniqueness of zero-continuation for elliptic equations, this yields that $\Phi = 0$ everywhere in M. As different ϕ_j, corresponding

to $\lambda_j = \lambda_{\widetilde{j}}$, are linearly independent, this implies that $(\psi, \phi_j)_{H^{-1}(M)\times H^1(M)} = 0$ for all $j = 1, 2, \dots$. Thus, $e = 0$ in $M \times \mathbb{R}$ and, therefore, $\psi = 0$. $\square$

4.2. **Approximate controllability with given trace at final time.** In this section we denote $\widetilde{\Sigma}$ to be either $\Sigma_i, i = 1, 2$, or Σ. Lemma 2.1 makes it possible to introduce a quasinorm

$$|h|^2 := \|Wh\|^2_{H^1(M)} = \sum_{j=1}^{\infty} (\lambda_j + 1)|u_j^{0,h}(0)|^2. \tag{4.13}$$

It is classical for the control theory, see e.g. [20] or [22] in the context of inverse problems, to define the space D^1 of the generalized sources by introducing the equivalence relation,

$$h \equiv_E \widetilde{h} \quad \text{if } u^{0,h}(0) = u^{0,\widetilde{h}}(0),$$

and completing $C_+^\infty(\Sigma \times \mathbb{R})/E$ with respect to (4.13),

$$D^1 := \mathrm{cl}\left(C_+^\infty(\Sigma \times \mathbb{R})/E\right).$$

Then, by Theorem 4.1, we can extend the wave operator W, see (4.2), from $C_+^\infty(\Sigma \times \mathbb{R})$ onto D^1,

$$Wh := u^{0,h}(0), \quad W : D^1 \to H^1(M),$$

as a unitary operator.

Moreover, as $Wh = \sum_{j=1}^\infty \kappa_j^h \phi_j \in H^1(M)$ and the Fourier coefficients κ_j^h, for any $h \in D^1$, can be explicitly evaluated using the Dirichlet spectral data (1.3), see the first formula in (2.10), it is possible to find, for such h ,the trace,

$$Wh|_\Sigma = \sum_{j=1}^\infty \kappa_j^h \phi_j|_\Sigma.$$

The above considerations give rise to the following lemma.

Lemma 4.2. *Assume* $\sigma(-\Delta^D(\overline{S})) \cap \sigma(-\Delta^D(M \setminus S)) = \emptyset$. *Then the subspaces,*

$$D^1_{\widetilde{\Sigma}} := \{h \in D^1 : Wh|_{\widetilde{\Sigma}} = 0\} \subset D^1,$$

are uniquely determined in terms of the Dirichlet spectral data (1.3).

Moreover, the wave operator W*, restricted to* $D^1_{\widetilde{\Sigma}}$*,*

$$W : D^1_{\widetilde{\Sigma}} \to H^1_{\widetilde{\Sigma}}, \quad H^1_{\widetilde{\Sigma}} := \{a \in H^1(M) : a|_{\widetilde{\Sigma}} = 0\},$$

is unitary.

4.3. **Finding eigenvalues and eigenfunctions in subdomains.** In this subsection we denote by $\widetilde{S}$ one of the manifolds S_i, $i = 1, 2$, $M \setminus S_i$, $i = 1, 2$, $M \setminus S$ and by $\lambda_n(\widetilde{S})$, $\phi_n(\cdot; \widetilde{S})$ we denote the eigenvalues and orthonormal eigenfunctions of $-\Delta^D(\widetilde{S})$. By the max-min principle,

$$\lambda_n(\widetilde{S}) = \max_{u_1,\dots,u_{n-1}} \min_{u_n} (\nabla_g u_n, \nabla_g u_n)_{L^2(\widetilde{S})},$$

where the maximum is taken over $u_1, u_2, \dots, u_{n-1} \in H^1_0(\widetilde{S})$ and the minimum is taken over $u_n \in H^1_0(\widetilde{S})$ that satisfies

$$(u_n, u_p)_{L^2(\widetilde{S})} = 0, \quad p = 1, 2, \dots, n-1, \quad (u_p, u_p)_{L^2(\widetilde{S})} = 1, \quad p = 1, 2, \dots, n.$$

The minimizer $u_n(x)$ is then an normalized eigenfunction corresponding to the eigenvalue $\lambda_n(\widetilde{S})$. Now consider the following max-min problem

$$t_n(\widetilde{\Sigma}) = \max_{u_1,\dots,u_{n-1}} \min_{u_n} (\nabla_g u_n, \nabla_g u_n)_{L^2(M)} = \max_{u_1,\dots,u_{n-1}} \min_{u_n} \sum_{j=1}^{\infty} \lambda_j |u_{n,j}|^2, \tag{4.14}$$

where $u_{n,j}$ are the Fourier coefficients of u_n, i.e. $u_n(x) = \sum_{j=1}^{\infty} u_{n,j}\phi_j(x)$. Here the maximum is taken over $u_1, u_2, \dots, u_{n-1} \in H^1(M)$ satisfying $u_p|_{\widetilde{\Sigma}} = 0$ and the minimum is taken over $u_n \in H^1(M)$ satisfuing $u_n|_{\widetilde{\Sigma}} = 0$ and

$$(u_n, u_p)_{L^2(M)} = \sum_{j=1}^{\infty} u_{n,j}\overline{u}_{p,j} = 0, \quad p = 1, 2, \dots, n-1, \tag{4.15}$$

$$(u_p, u_p)_{L^2(M)} = \sum_{j=1}^{\infty} |u_{p,j}|^2 = 1, \quad p = 1, 2, \dots, n.$$

Then $t_n(\widetilde{\Sigma})$ are the eigenvalues of the Dirichlet Laplacian on the direct sum of $L^2(\widetilde{S})$ and $L^2(M \setminus \widetilde{S})$, so that

$$\{t_n(\widetilde{\Sigma})\}_{n=1}^{\infty} = \sigma(-\Delta^D(\widetilde{S})) \cup \sigma(-\Delta^D(M \setminus \widetilde{S})).$$

The sequence of the corresponding minimizers, $u_n(x; \widetilde{\Sigma})$ consists of orthonormal eigenfunctions of this operator. However, due to the assumption $\sigma(-\Delta^D(\widetilde{S})) \cap \sigma(-\Delta^D(M \setminus \widetilde{S})) = \emptyset$, any such eigenfunction is equal to 0 on $M \setminus \widetilde{S}$ or $\widetilde{S}$. Thus, any $u_n(x; \widetilde{\Sigma})$ is either an eigenfunction of $-\Delta^D(\widetilde{S})$ extended by 0 to $M \setminus \widetilde{S}$, or an eigenfunction of $-\Delta^D(M \setminus \widetilde{S})$ extended by 0 to $\widetilde{S}$.

On the other hand, Lemma 4.2 together with equation (2.10) make it possible to evaluate the right-hand sides in (4.14), (4.15) using the Dirichlet spectral data. This leads to the following result

Lemma 4.3. *Let $\sigma(-\Delta^D(\overline{S_i}))$, $\sigma(-\Delta^D(M \setminus S_i)), i = 1, 2$ and $\sigma(-\Delta^D(M \setminus S))$ be all disjoint. Then the Dirichlet spectral data (1.3) determine uniquely the*

eigenvalues $\lambda_n(\widetilde{S})$, $n = 1, 2, \dots$, $\widetilde{S} = S, S_i, M \setminus S_i$, $i = 1, 2$, *and* $M \setminus S$. *They determine also the the generalized sources* $h_n(\widetilde{S})$ *such that*

$$Wh_n(\widetilde{S}) = u^{0,h_n(\widetilde{S})}(x,0) = \begin{cases} \phi_n(x;\, \widetilde{S}), & x \in \widetilde{S}, \\ 0, & x \in M \setminus \widetilde{S}. \end{cases} \tag{4.16}$$

In addition, the Dirichlet spectral data determine the Fourier coefficients of the extended eigenfunctions $Wh_n(\widetilde{S})(x)$,

$$Wh_n(\widetilde{S}) = \sum_{j=1}^{\infty} \kappa_{n,j}(\widetilde{S})\phi_j(x), \quad x \in M. \tag{4.17}$$

We note that these sources $h_n(\widetilde{S})$ are determined up to a unitary transformation in the eigenspace corresponding to the eigenvalue $\lambda_n(\widetilde{S})$.

Proof. Recall that, by formula (2.10) we can evaluate the Fourier coefficients $u_j^{0,h}(0)$ for any $h \in D^1$. Thus, we can evaluate

$$\mu_n(\widetilde{\Sigma}) = \max_{h_1,\dots,h_{n-1}} \min_{h_n} \sum_{j=1}^{\infty} \lambda_j |u_j^{0,h_n}(0)|^2,$$

where the maximum is taken over $h_1, h_2, \dots, h_{n-1} \in D^1_{\widetilde{\Sigma}}$ and the minimum is taken over $h_n \in D^1_{\widetilde{\Sigma}}$ with

$$\sum_{j=1}^{\infty} u_j^{0,h_n}(0)\overline{u}_j^{0,h_p}(0) = 0,\ p = 1, \dots, n-1, \quad \sum_{j=1}^{\infty} |u_j^{0,h_p}(0)|^2 = 1,\ p = 1, \dots, n.$$

It follows from Lemma 4.2 that, for any $n = 1, 2, \dots$, $\mu_n(\widetilde{\Sigma}) = t_n(\widetilde{\Sigma})$, providing $\sigma(-\Delta^D(\overline{\widetilde{S}})) \cap \sigma(-\Delta^D(M \setminus \widetilde{S})) = \emptyset$.

Repeating this construction with $\widetilde{\Sigma}$ equal to Σ, Σ_1, and Σ_2, we obtain the sets

$\sigma(-\Delta^D(S)) \cup \sigma(-\Delta^D(M \setminus S)) = \sigma(-\Delta^D(S_1)) \cup \sigma(-\Delta^D(S_2)) \cup \sigma(-\Delta^D(M \setminus S))$,

and

$$\sigma(-\Delta^D(S_1)) \cup \sigma(-\Delta^D(M \setminus S_1)), \qquad \sigma(-\Delta^D(S_2)) \cup \sigma(-\Delta^D(M \setminus S_2)).$$

As $\sigma(-\Delta^D(\overline{S_i}))$, $\sigma(-\Delta^D(M \setminus S_i)), i = 1, 2$ and $\sigma(-\Delta^D(M \setminus S))$ are all disjoint, by intersecting the above sets we find the desired eigenvalues $\lambda_n(S_i)$, $n = 1, 2, \dots, i = 1, 2$ and $\lambda_n(M \setminus S_i)$, $n = 1, 2, \dots, i = 1, 2$, as well as $\lambda_n(M \setminus S)$, $n = 1, 2, \dots$.

Then, identifying the corresponding subsequence of $t_n(\widetilde{\Sigma})$ and related generalized sources $h_n(\widetilde{\Sigma})$, we determine, for each $\widetilde{S}$, the generalized sources $h_n(\widetilde{S})$ such that $Wh_n(\widetilde{S})$ are equal to the extended eigenfunctions (4.16).

Recalling formula (2.10), we prove the last part of the Lemma. □

4.4. **Inverse problems in subdomains. Proof of Theorem 1.2.** Our proof of Theorem 1.2 is based on Lemma 4.3. Namely, we will show that, having at hand the eigenvalues $\lambda_n(S_i)$, $i = 1, 2$, and $\lambda_n(M \setminus S)$ and also the Fourier coefficients, $\kappa_{n,j}(S_i)$, $i = 1, 2$, $\kappa_{n,j}(M \setminus S)$, it is possible to determine, up to an isometry, the Riemannian manifolds (S_i, g), $i = 1, 2$, $(M \setminus S, g)$. Gluing them along Σ_i we recover (M, g).

Recall that if, for (N, g), $\partial N \neq \emptyset$, we do know its Dirichlet eigenvalues $\lambda_n(N)$ and traces on ∂N of the normal derivatives of the eigenfunctions, $\partial_\nu \psi_n|_{\partial N}$, then these data determine (N, g) up to an isometry, see e.g. [17]. However, in the case of the Dirichlet spectral data, we have only the Dirichlet values $\phi_j|_\Sigma$ and, moreover, the convergence of the Fourier series (4.17) is only in $H^1(M)$ preventing us from identifying $\partial_\nu \phi_n(\widetilde{S})|_{\widetilde{\Sigma}}$. Therefore, we will use another approach within the BC-method, described in section 4.1 of [17]. To explain it, consider the intial-boundary value problem in $N \times \mathbb{R}$,

$$\begin{aligned} &(\partial_t^2 - \Delta_g) w_F = 0, \quad \text{in } N \times \mathbb{R}_+, \\ &w_F|_{\partial N \times \mathbb{R}_+} = F, \quad w_F|_{t=0} = 0,\, \partial_t w_F|_{t=0} = 0, \end{aligned} \tag{4.18}$$

with $F \in C_+^\infty(\partial N \times \mathbb{R}_+)$. The energy, at time t, of the wave w_F is then defined as

$$E(w_F, t) = \frac{1}{2} \int_N (|\partial_t w_F(x,t)|^2 + |\nabla_g w_F(x,t)|_g^2) dV_g(x).$$

It is shown in [17] that, given the energy flux

$$\Pi(F) := \lim_{t \to \infty} E(w_F, t),$$

for any $F \in C_0^\infty(\partial N \times \mathbb{R}_+)$, it is possible to determine (N, g) up to an isometry.

Therefore, Theorem 1.2 is an immediate corollary of the following Lemma,

Lemma 4.4. *Let $\Sigma = \Sigma_1 \cup \Sigma_2$ divide M into regions S_i, $i = 1, 2$, and $M \setminus S$. Consider the initial-boundary value problems (4.18) with N equal to $\overline{S}_1$, $\overline{S}_2$ and $M \setminus S$. Then, assuming that $\sigma(-\Delta^D(\overline{S_i}))$, $\sigma(-\Delta^D(M \setminus S_i))$, $i = 1, 2$, and $\sigma(-\Delta^D(M \setminus S))$ are all disjoint, the Dirichlet spectral data (1.3) uniquely determine the energy flux $\Pi(F)$ in each of these subdomains.*

Proof. We start with the case $N = \overline{S}_1$. Then (4.18) takes the form

$$\begin{aligned} &(\partial_t^2 - \Delta_g) w_F = 0, \quad \text{in } S_1 \times \mathbb{R}_+, \\ &w_F|_{\Sigma_1 \times \mathbb{R}_+} = F, \quad w_F|_{t=0} = 0,\, \partial_t w_F|_{t=0} = 0, \end{aligned} \tag{4.19}$$

with $F \in C_+^\infty(\Sigma_1 \times \mathbb{R}_+)$. Let us first show that, for any such F, there exists a unique $h_F \in C_+^\infty(\Sigma_1 \times \mathbb{R}_+)$ such that $w_F = u^{0,h}|_{S_1}$, where

$$h = \begin{cases} h_F, & x \in \Sigma_1, \\ 0, & x \in \Sigma_2. \end{cases} \tag{4.20}$$

To this end we consider, in addition to (4.19), the problem

$$\begin{aligned}&(\partial_t^2-\Delta_g)w_F^c=0, \quad \text{in } (M\setminus S_1)\times\mathbb{R}_+,\\ &w_F^c|_{\Sigma_1\times\mathbb{R}_+}=F, \quad w_F^c|_{t=0}=0,\ \partial_t w_F^c|_{t=0}=0,\end{aligned}$$

and introduce the function u,

$$u=\begin{cases} w_F, & x\in S_1\times\mathbb{R}_+,\\ w_F^c, & x\in (M\setminus S_1)\times\mathbb{R}_+.\end{cases} \tag{4.21}$$

Then u solves the transmission problem (2.8) with $f=0$ and

$$h=\begin{cases} h_F=(\partial_\nu w_F^c-\partial_\nu w_F)\,|_{\Sigma_1\times\mathbb{R}_+}, & (x,t)\in\Sigma_1\times\mathbb{R}_+\\ 0, & (x,t)\in\Sigma_2\times\mathbb{R}_+,\end{cases}$$

i.e. $u=u^{0,h}$. Using considerations similar to those in the proof of Lemma 3.3, we show the uniqueness of such h.

By Corollary 2.2, we can then find, for $h\in C_+^\infty(\Sigma_1\times\mathbb{R}_+)$,

$$\Lambda_1 h=u^{0,h}|_{\Sigma_1\times\mathbb{R}_+}. \tag{4.22}$$

As shown earlier, for $h=0$ on Σ_2, the operator Λ_1 is an invertible operator in $C_+^\infty(\Sigma_1\times\mathbb{R}_+)$. Thus, we can use equation (4.22) with the right-hand side being F, to uniquely determine h_F. Observe that the extended eigenfunctions $\{W\left(h_n(S_1)\right)\}_{n=1}^\infty$ together with $\{W\left(h_k(M\setminus S_1)\right)\}_{k=1}^\infty$ form an orthonormal basis in $L^2(M)$. Thus, using (4.17), we can evaluate the Fourier coefficients of $u^{0,h}(\cdot,t)$ with respect to this basis,

$$u^{0,h}(x,t)=\sum_{n=1}^\infty w_{n,F}(t)\,W\left(h_n(S_1)\right)(x)+\sum_{k=1}^\infty w_{k,F}^c(t)\,W\left(h_k(M\setminus S_1)\right)(x),$$

where the index F indicates that h is of form (4.20). This expansion, together with the definition of u, see (4.21), shows that

$$\begin{aligned}&w_F(x,t)=\Theta_1(x)u^{0,h}(x,t)=\sum_{n=1}^\infty w_{n,F}(t)\,W\left(h_n(S_1)\right)(x),\\ &\|w_F(t)\|^2_{L^2(S_1)}=\sum_{n=1}^\infty |w_{n,F}(t)|^2,\end{aligned} \tag{4.23}$$

where Θ_1 is the characteristic function of S_1 and we identify functions in S_1 with their extensions by 0 to $M\setminus S_1$.

Similarly,

$$\partial_t w_F(x,t)=\sum_{n=1}^\infty \partial_t w_{n,F}(t)\,W\left(h_n(S_1)\right)(x), \quad \|\partial_t w_F(t)\|^2_{L^2(S_1)}=\sum_{n=1}^\infty |\partial_t w_{n,F}(t)|^2$$

The last step of the construction is based on the observation that, when $F \in C_0^\infty(\Sigma_1 \times \mathbb{R}_+)$, then $w_F \in H_0^1(S_1)$ for large t and, therefore, the eigenfunction expansions (4.23) converge in $H^1(S_1)$. Thus, we can find

$$\|\nabla_g w_F(\cdot, t)\|^2_{L^2(S_1)} = \sum_{n=1}^{\infty} \lambda_n(S_1)|w_{n,F}(t)|^2.$$

Combining the above two equations, we find the energy flux $\Pi(F)$.

Similar considerations, with $\overline{S}_2$ and $M \setminus S_2$ and $M \setminus S$ and $\overline{S}$ instead of $\overline{S}_1$ and $M \setminus S_1$, show the possibility to evaluate the energy flux when $N = \overline{S}_2$ and $M \setminus S$. □

5. Some remarks on further generalizations and open problems

Remark 5.1. The inverse problem with the Cauchy or Dirichlet spectral data on a compact Riemannian manifold without boundary studied in Sections 3, 4 can be generalized to the problem when M has boundary $\partial M \neq \emptyset$ with e.g. Neumann (or Dirichlet) boundary condition. Then (1.2) and (1.3) consists of the Neumann eigenvalues of M and the Cauchy or Dirichlet data of the eigenfuctions on a closed hypersurface $\Sigma \subset M$, $\partial M \cap \Sigma = \emptyset$. The methods to solve these problems are very similar to the described and are based on the version of the BC-method with data on a part of boundary, see section 4.4 [17].

Remark 5.2. Consider the non-stationary Green function, $G(x, y; t)$, for the acoustic wave equation in (M, g), where M may have a non-trivial boundary,

$$(\partial_t^2 - \Delta_g)G(x, y; t) = \delta_y(x)\delta(t), \quad \text{in } M \times \mathbb{R},$$
$$G(x, y; t)|_{\partial M \times \mathbb{R}_+} = 0, \quad G(x, y; t)|_{t<0} = 0,$$

where $y \in \Sigma$, $\Sigma \subset M$ such that $\Sigma \cap \partial M = \emptyset$ and the boundary condition is void when $\partial M = \emptyset$. It often happens in practice, for example in geophysics or ultrasound imaging, that one can measure $G(x, y; t)$ for x again running only over Σ. Thus, we come to the inverse problem with non-stationary data being $G(x, y; t)$, $x, y \in \Sigma$, $t > 0$.

Taking the Fourier transform of the given $G(x, y; t)$ in t, we find the Green function $G(x, y; k)$, cf. [15],

$$(-k^2 - \Delta_g)G(x, y; k) = \delta_y(x), \quad G(x, y; t)|_{\partial M} = 0.$$

Thus, from practical measurements we can find $G(x, y; k)$ for $x, y \in \Sigma$, $k \in \mathbb{R}$.

Note that $G(x, y; k)$, $x, y \in \Sigma$, is the integral kernel of a meromorphic, with respect to $k \in \mathbb{C}$, operator-valued function in $L^2(\Sigma)$. In terms of the eigenvalues

and eigenfunctions of $-\Delta_g$,

$$G(x,y;k)|_{\Sigma\times\Sigma} = \sum_{j=1}^{\infty} \frac{1}{k^2-\lambda_j}\phi_j(x)|_\Sigma \phi_j(y)|_\Sigma,$$

where the convergence in the right-hand side is understood in the sense of operators in $L^2(\Sigma)$. So finding the poles of $G(x,y;k)$ is equivalent to determination of $\sqrt{\lambda_j}$. At the pole $\sqrt{\lambda_j}$, the residue is given by the integral operator with the kernel,

$$\mathrm{Res}(G(x,y;\cdot),\sqrt{\lambda_j}) = \frac{1}{2\sqrt{\lambda_j}} \sum_{l:\lambda_l=\lambda_j} \phi_l(x)|_\Sigma \phi_l(y)|_\Sigma.$$

The knowledge of this kernel allows us to find the functions $\phi_j|_\Sigma$ up to an orthogonal transformation in the eigenspace corresponding to λ_j, see [15], [31].

Therefore, the dynamic inverse data $G(x,y;t)$, $x,y\in\Sigma$, $t>0$ makes it possible to find the Dirichlet spectral data on Σ.

Remark 5.3. As shown in section 4, the Dirichlet data (1.3) determine (M,g) when the spectra $\sigma(-\Delta^D(\overline{S_i}))$, $\sigma(-\Delta^D(M\setminus S_i)), i=1,2$, and $\sigma(-\Delta^D(M\setminus S))$ are all disjoint. It is interesting to understand whether this condition can be removed.

In general, it is important to find if the Dirichlet spectral data (1.3) determine (M,g) even in the case when Σ is connected.

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